Some reciprocal summation identities with applications to the Fibonacci and Lucas numbers

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1: Introduction

In this arcticle we use theta functions and simple telescoping of series to produce some reciprocal summation results for the Fibonacci and Lucas numbers. The two results that we prove are the following:

Theorem 1

$$\left(\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}}\right)^2 - \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^2} = 2\sum_{n=1}^{\infty} \frac{1}{F_{4n-2}^2}$$

Theorem 2

$$\sum_{n=1}^{\infty} \left(\frac{2^n}{L_{2^n}}\right)^2 = \frac{4}{5}$$

1F_n and 1L_n are the Fibonacci and Lucas numbers respectively, satisfying the usual recurrence 1 $U_{n+1} = U_n + U_{n-1}$ where 1 $F_0 = 0$, 1 $F_1 = 1$, 1 $L_0 = 2$ and
1 $L_1 = 1$.

2: Proof of Theorem 1

By simple series rearrangement we have

$$\sum_{n=1}^{\infty} \frac{q^{2n-1}}{(1+q^{2n-1})^2} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{nq^n}{1-q^{2n}}.$$

Replacing 2 q by 2 q^2 in the above then splitting the summation over the odd and even numbers we obtain

$$\begin{aligned} 2\sum_{n=1}^{\infty} \frac{q^{4n-2}}{(1+q^{4n-2})^2} \qquad 2 \qquad &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{nq^{2n}}{1-q^{4n}} \\ 2 \qquad 2 &= \sum_{n=1}^{\infty} (2n-1) \frac{q^{4n-2}}{1-q^{8n-4}} - 2\sum_{n=1}^{\infty} \frac{nq^{4n}}{1-q^{8n}}. \end{aligned} \qquad 2(1)$$

Now a theorem originally due to Guass gives

$$\sum_{n=1}^{\infty} (2n-1) \frac{q^{8n-4}}{1-q^{16n-8}} = \left\{ \frac{1}{2} \theta_2(q^4) \right\}^4$$

where

$$\theta_2(q) = \sum_{-\infty}^{\infty} q^{(n+1/2)^2} = 2q^{1/4} \prod_{n=1}^{\infty} (1-q^{2n})(1+q^{2n})^2.$$

Therefore

$$\sum_{n=1}^{\infty} (2n-1) \frac{q^{4n-2}}{1-q^{8n-4}} = \left(\sum_{n=1}^{\infty} \frac{q^{2n-1}}{1+q^{4n-2}} \right)^2, \qquad 2(2)$$

using

$$\theta_2(q^2)^2 = 4\sum_{n=1}^{\infty} \frac{q^{2n-1}}{1+q^{4n-2}}.$$

The last equality follows directly from equating the coefficient of 2 x in (2) of **[1]**.
Also, by simple series rearrangement we have

$$\sum_{n=1}^{\infty} \frac{nq^n}{1-q^{2n}} = \sum_{n=1}^{\infty} \frac{q^{2n-1}}{(1-q^{2n-1})^2},$$

which on replacing 2 q by 2 q^4 gives

$$\sum_{n=1}^{\infty} \frac{nq^{4n}}{1-q^{8n}} = \sum_{n=1}^{\infty} \frac{q^{8n-4}}{(1-q^{8n-4})^2}. \qquad 2(3)$$

Using (2) and (3) to substitute for the summations in the RHS of (1) gives

$$\sum_{n=1}^{\infty} \frac{q^{4n-2}}{(1+q^{4n-2})^2} = \left(\sum_{n=1}^{\infty} \frac{q^{2n-1}}{1+q^{4n-2}} \right)^2 - 2 \sum_{n=1}^{\infty} \frac{q^{8n-4}}{(1-q^{8n-4})^2}. \qquad 2(4)$$

Setting 3 $q = (1-\sqrt{5})/2$ in (4), and using the Binet form 3 $F_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ of the Fibonacci numbers (3 $\alpha = (1+\sqrt{5})/2$, 3 $\beta = (1-\sqrt{5})/2$) we obtain Theorem 1.

3: Proof of Theorem 2

Our starting point is the identity

$$\frac{q}{(1+q)^2} + \frac{4q^2}{(1-q^2)^2} = \frac{q}{(1-q)^2}.$$

This telescopes to

$$\frac{q}{(1+q)^2} + \frac{4q^2}{(1+q^2)^2} + \frac{16q^4}{(1-q^4)^2} = \frac{q}{(1-q)^2}$$

and continuing the expansion process we arrive at

$$\sum_{n=0}^{\infty} \frac{2^{2n} q^{2^n}}{(1+q^{2^n})^2} = \frac{q}{(1-q)^2}.$$

Now we put 3 $q = (1-\sqrt{5})/2$ in the above identity and use the Binet form of 3 $L_n = \alpha^n + \beta^n$ of the Lucas numbers (3 α, 3 β as in section 2) to obtain Theorem 2.

References

[1] D.Jennings *An identity for the Fibonacci and Lucas numbers* Glasgow Math. J. **35** (1993) 381–384.